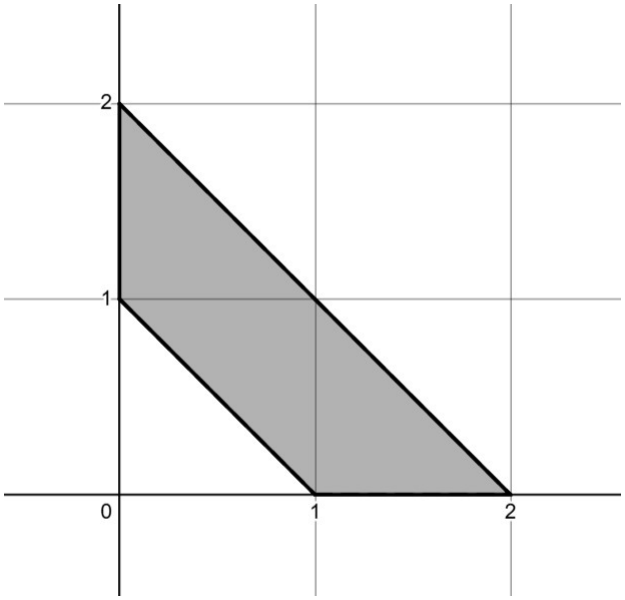
# Section 6.4

**Another way to cut solids**

Let us revisit a volume problem from the previous section:

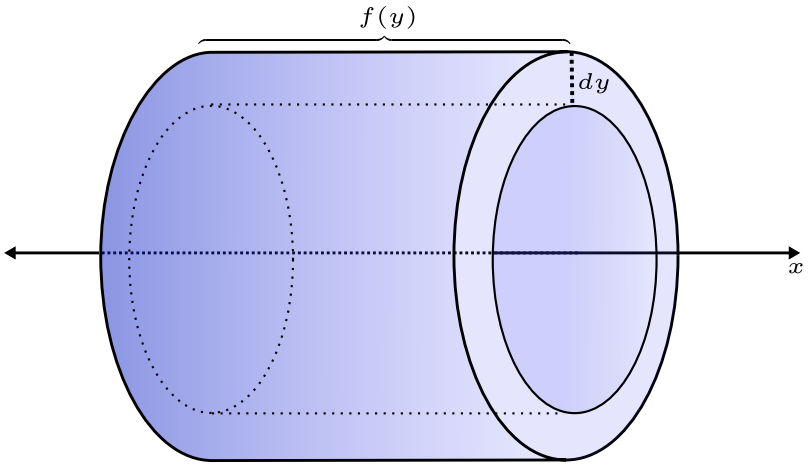
Let $R$ be the region bounded by the curves $y = 2 - x$, $y = 1 - x$, $x = 0$, and $y = 0$. Find the volume of the solid of revolution formed by revolving $R$ about the $x$-axis.




In the above integral, we swept through the solid along the $x$-axis and hence integrated with respect to $x$. This forced our integrand to be a function of $x$, i.e. $f(x)^2 - g(x)^2$.

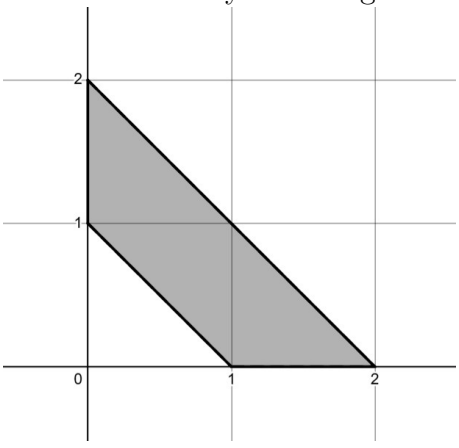
If we may want to move along the $y$-axis instead. Doing so creates three changes: the differential will become $dy$, the integrand must be a function of $y$ so that we may know how to successfully integrate, the solid is no longer divided into washer shaped chunks.

For a fixed value $y$, we can look at all points in the solid that have a distance from the axis approximately $y$. This creates a thin cylinder or *shell* inside our solid.




The shell has length $f(y)$, radius our chosen fixed $y$, and thickness $dy$. The volume of this shell is

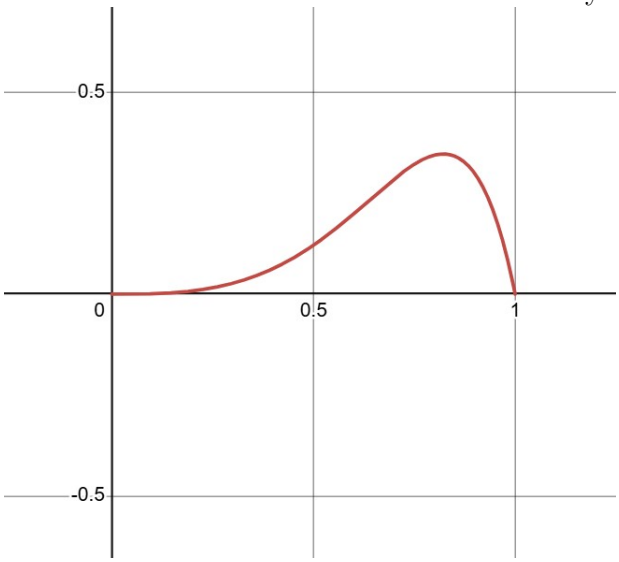
We'll use these shells to re-solve the previous example. Let $R$ be the region bounded by the curves $y = 2 - x$, $y = 1 - x$, $x = 0$, and $y = 0$. Find the volume of the solid of revolution formed by revolving $R$ about the $x$-axis.



**Why have two methods?**

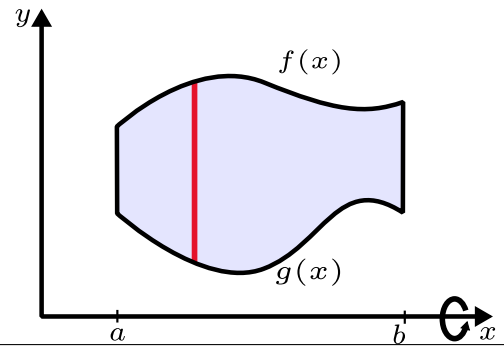
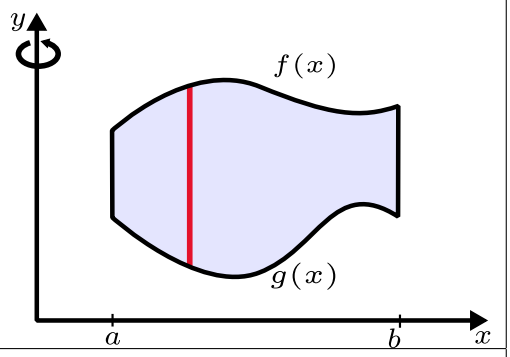
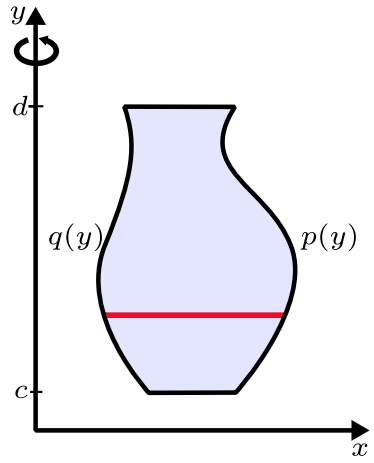
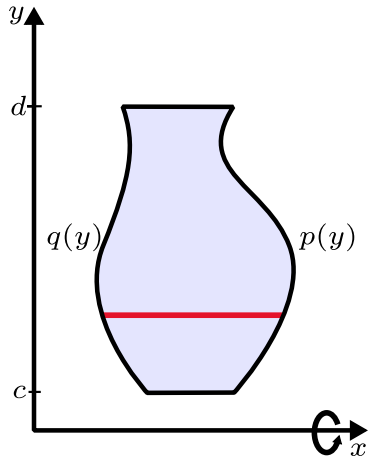
There are some solids of revolution that we would have no way to write the radii for the washer as functions in the appropriate variable.

**Example 6.4.1.** Let $R$ be the region bounded by $y = x^3 - x^8$ and the $x$-axis. Find the volume of the solid of revolution formed by revolving $R$ about the $y$-axis.

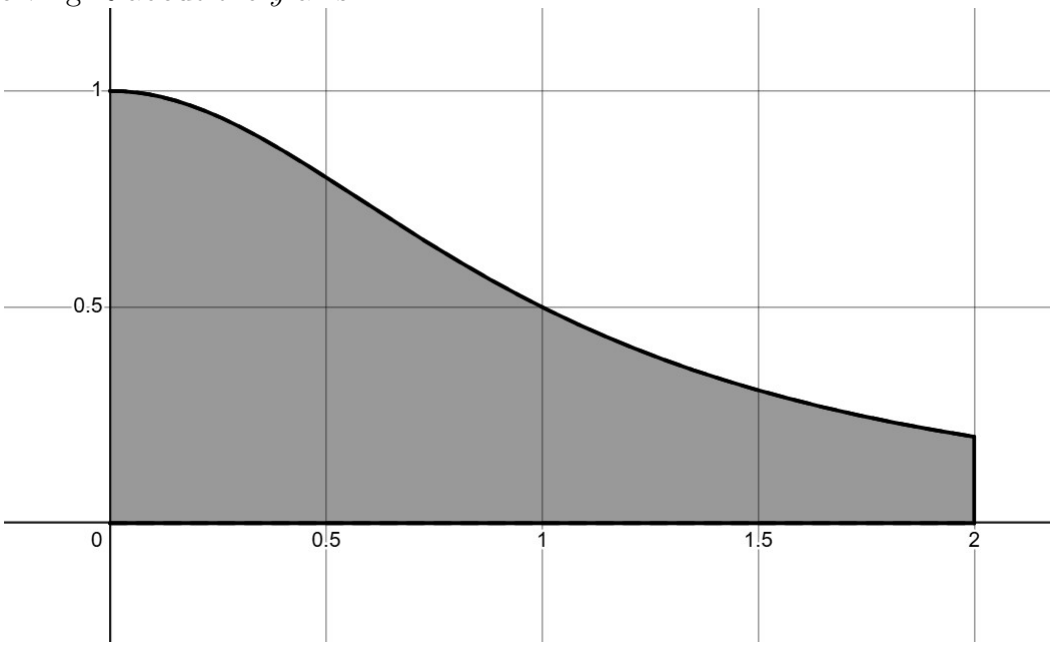


**How do we know which method to use?**

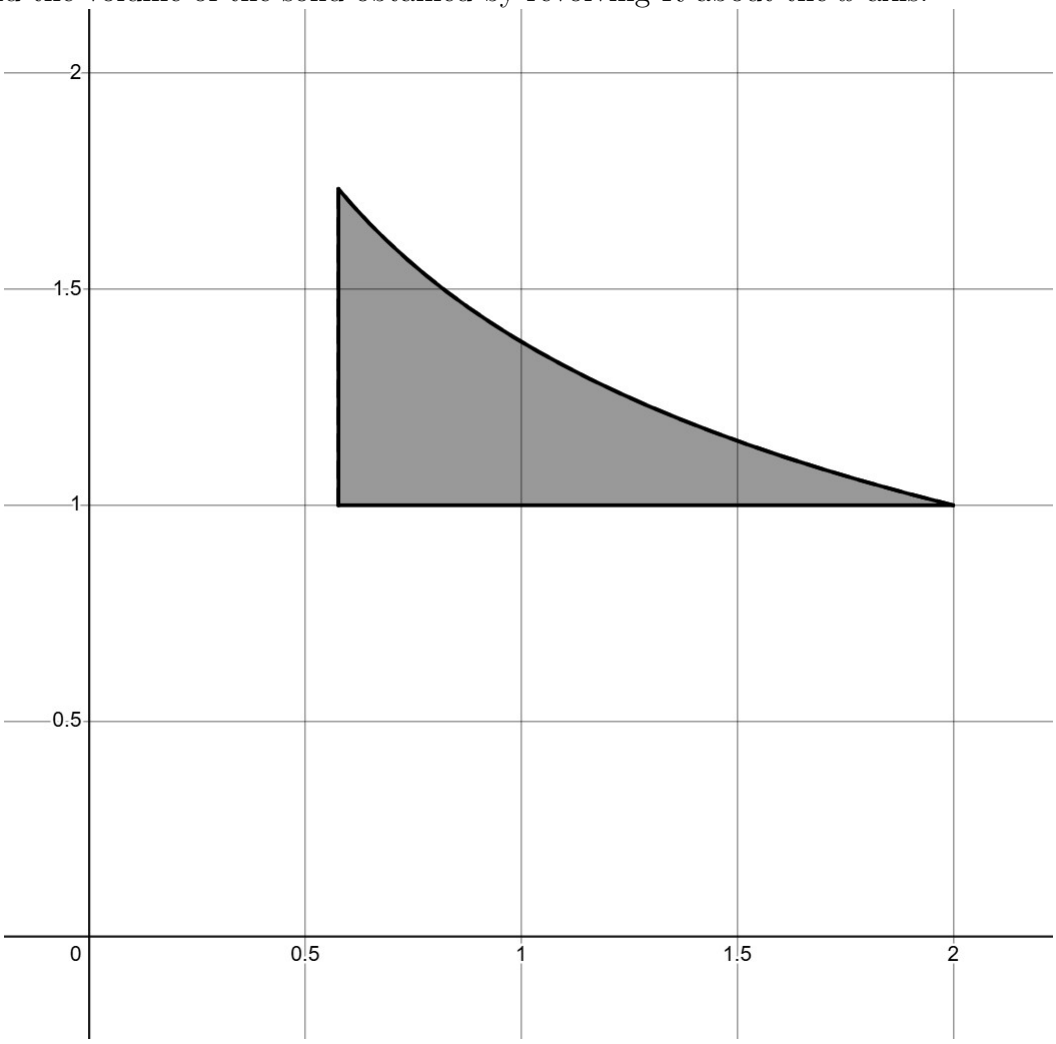
In principle, both methods correctly compute the volume of a solid of revolution. In practice, one method is usually easier to implement than the other for a given problem. One way to answer this is to ask yourself: "If I were to find the area of the region $R$, would I integrate with respect to $x$ or with respect to $y$?" Whichever the answer to this question, you will integrate the volume integral with respect to the same variable:

<table>
<tr>
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<tr>
<td>Washers are perpendicular to $x$-axis.<br>$\int_a^b \pi(f(x)^2 - g(x)^2)dx$</td>
<td>Shells are parallel to $y$-axis.<br>$\int_a^b 2\pi x(f(x) - g(x))dx$</td>
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<tr>
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<tr>
<td>Washers are perpendicular to the $y$-axis.<br>$\int_c^d \pi(p(y)^2 - q(y)^2)dy$</td>
<td>Shells are parallel to the $x$-axis.<br>$\int_c^d 2\pi y(p(y) - q(y))dy$</td>
</tr>
</table>

**Example 6.4.2.** Let $R$ be the region bounded by the curves $y = \dfrac{1}{1+x^2}$, $y = 0$, $x = 0$, and $x = 2$. Use the shell method to find the volume of the solid obtained by revolving $R$ about the $y$-axis.



**Example 6.4.3.** Let $R$ be the region bounded by $x = \dfrac{4}{y+y^3}$, $x = \dfrac{1}{\sqrt{3}}$, and $y = 1$. Find the volume of the solid obtained by revolving $R$ about the $x$-axis.



**Example 6.4.4.** Find the volume of the solid of revolution obtained by revolving the region $R$ from Example 6.4.3 about the line $y = 1$.